MATHMATICS III

1.3 exponents: Exponentiation is a mathematical operation, written as bn, involving two numbers, the base b and the exponent (or power) n. When n is a positive integer, exponentiation corresponds to repeated multiplication; in other words, a product of n factors, each of which is equal to b (the product itself can also be called power):

The exponent is usually shown as a superscript to the right of the base. The exponentiation bn can be read as: b raised to the n-th power, b raised to the power of n, or b raised by the exponent of n, most briefly as b to the n. Some exponents have their own pronunciation: for example, b2 is usually read as b squared and b3 as b cubed.

The power bn can be defined also when n is a negative integer, for nonzero b. No natural extension to all real b and n exists, but when the base b is a positive real number, bn can be defined for all real and even complex exponents n via the exponential function ez. Trigonometric functions can be expressed in terms of complex exponentiation.

Exponentiation, where the exponent is a matrix, is used for solving systems of linear differential equations.

Exponentiation is used pervasively in many other fields, including economics, biology, chemistry, physics, as well as computer science, with applications such as compound interest, population growth, chemical reaction kinetics, wave behavior, and public-key cryptography.

Background and terminology

The expression b2 = b·b is called the square of b because the area of a square with side-length b is b2. It is pronounced "b squared".

The expression b3 = b·b·b is called the cube of b because the volume of a cube with side-length b is b3. It is pronounced "b cubed".

The exponent says how many copies of the base are multiplied together. For example, 35 = 3·3·3·3·3 = 243. The base 3 appears 5 times in the repeated multiplication, because the exponent is 5. Here, 3 is the base, 5 is the exponent, and 243 is the power or, more specifically, the fifth power of 3, 3 raised to the fifth power, or 3 to the power of 5.

The word "raised" is usually omitted, and very often "power" as well, so 35 is typically pronounced "three to the fifth" or "three to the five".

Exponentiation may be generalized from integer exponents to more general types of numbers. The word "exponent" was coined in 1544 by Michael Stifel. The modern notation for exponentiation was introduced by René Descartes in his Géométrie of 1637.

Particular bases[edit]
Powers of ten[edit]
See also: Scientific notation
In the base ten (decimal) number system, integer powers of 10 are written as the digit 1 followed or preceded by a number of zeroes determined by the sign and magnitude of the exponent. For example, 103 = 1,000 and 10−4 = 0.0001.

Exponentiation with base 10 is used in scientific notation to denote large or small numbers. For instance, 299,792,458 m/s (the speed of light in vacuum, in metre per second) can be written as 2.99792458×108 m/s and then approximated as 2.998×108 m/s.

SI prefixes based on powers of 10 are also used to describe small or large quantities. For example, the prefix kilo means 103 = 1,000, so a kilometre is 1,000 metres.

Powers of two
The positive powers of 2 are important in computer science because there are 2n possible values for an n-bit binary register.

Powers of 2 are important in set theory since a set with n members has a power set, or set of all subsets of the original set, with 2n members.

The negative powers of 2 are commonly used, and the first two have special names: half, and quarter.

In the base 2 (binary) number system, integer powers of 2 are written as 1 followed or preceded by a number of zeroes determined by the sign and magnitude of the exponent. For example, two to the power of three is written as 1000 in binary.

Powers of one

The integer powers of one are all one: 1n = 1.

Powers of zero
If the exponent is positive, the power of zero is zero: 0n = 0, where n > 0.

If the exponent is negative, the power of zero (0n, where n < 0) is undefined, because division by zero is implied.

If the exponent is zero, some authors define 00 = 1, whereas others leave it undefined, as discussed below.

Powers of minus one
If n is an even integer, then (−1)n = 1.

If n is an odd integer, then (−1)n = −1.

Because of this, powers of −1 are useful for expressing alternating sequences. For a similar discussion of powers of the complex number i, see the section on Powers of complex numbers.

Large exponents
The limit of a sequence of powers of a number greater than one diverges, in other words they grow without bound:

bn → ∞ as n → ∞ when b > 1
This can be read as "b to the power of n tends to +∞ as n tends to infinity when b is greater than one".

Powers of a number with absolute value less than one tend to zero:

bn → 0 as n → ∞ when |b| < 1
Any power of one is always itself:

bn = 1 for all n if b = 1
If the number b varies tending to 1 as the exponent tends to infinity then the limit is not necessarily one of those above. A particularly important case is

(1 + 1/n)n → e as n → ∞
See the section below, The exponential function.

Other limits, in particular of those tending to indeterminate forms, are described in limits of powers below.

1.3.1 Laws: commutative law, in mathematics, either of two laws relating to number operations of addition andmultiplication, stated symbolically: $a + b = b + a$ and $ab = ba$. From these laws it follows that any finite sum or product is unaltered by reordering its terms or factors. While commutativity holds for many systems, such as the real or complex numbers, there are other systems, such as the system of $n \times n$matrices or the system of quaternions, in which commutativity of multiplication is invalid. Scalar multiplication of two vectors (to give the so-called dot product) is commutative (i.e., $a \cdot b = b \cdot a$), but vector multiplication (to give the cross product) is not (i.e., $a \times b = -b \times a$). The commutative law does not necessarily hold for multiplication of conditionally convergent series.

In many mathematical domains it is useful to join two objects in some way to create a new object. Thus out of two numbers one may construct their sum or product, from two sets one may form the union or intersections, and from two propositions one may form a new one using AND or OR. Such practices are not limited to mathematics, for in language we link words such as “race” and “horse” to form “racehorse” or join clauses to form a compound sentence. In mathematics, objects are linked by operations. Many properties of operations play an important role, the three most prominent being the associative, commutative, and distributive laws.

associative law, in mathematics, either of two laws relating to number operations of addition and multiplication, stated symbolically: $a + (b + c) = (a + b) + c$, and $a(bc) = (ab)c$; that is, the terms or factors may be associated in any way desired. While associativity holds for ordinary arithmetic with real or imaginary numbers, there are certain applications—such as nonassociative algebras—in which it does not hold.

In mathematics, the associative property is a property of some binary operations. In propositional logic, associativity is a valid rule of replacement for expressions in logical proofs.

Within an expression containing two or more occurrences in a row of the same associative operator, the order in which the operations are performed does not matter as long as the sequence of the operands is not changed. That is, rearranging the parentheses in such an expression will not change its value.

Even though the parentheses were rearranged, the values of the expressions were not altered. Since this holds true when performing addition and multiplication on any real numbers, it can be said that "addition and multiplication of real numbers are associative operations."

Associativity is not to be confused with commutativity, which addresses whether $a \times b = b \times a$.

Associative operations are abundant in mathematics; in fact, many algebraic structures (such as semigroups and categories) explicitly require their binary operations to be associative.

However, many important and interesting operations are non-associative; some examples include subtraction, exponentiation and the vector cross product. In contrast to the theoretical counterpart, the addition of floating point numbers in computer science is not associative, and is an important source of rounding error.

When an operation is defined, it is at once clear how two objects are to be combined. But what if we have three objects? If we call these objects a, b, c, and we wish to combine them in the given order, then there are two possibilities: We can first operate on a and b and combine the result with c, or we can combine a with the result of operating first on b and c.

Let us take, for example, the operation of addition and the numbers 4, 17, and 2. We may then interpret the expression $4 + 17 + 2$ either as $(4 + 17) + 2$ or as $4 + (17 + 2)$. In the first case, we get $21 + 2$, and in the second case we get $4 + 19$. It turns out that in both cases we get the final result 23.
Operations for which—as in the case of addition—both arrangements always lead to the same result are called associative, and we say that for that operation, the associative law holds.

Some Important Points
1. Most operations that one encounters are associative: addition, multiplication, union and intersection of sets, logical operations such as AND and OR, and so on. However, it is not difficult to find operations that violate the distributive law. Language is certainly not associative. A newspaper headline of many years ago read, “Rail Service for Area Dead.” Was the local railway service defunct—(Rail

Service for Area) Dead—or was there a new rail service for transporting local corpses—Rail Service for (Area Dead)? And was the “purple PEOPLE eater” of the 1958 hit song a purple creature that eats people—purple (people eater)—or a creature that eats purple people—(purple people) eater?

In another newspaper article, one could read that “girls and boys from homes with educated parents are given more encouragement by their teachers. It became clear in the course of the article that what was meant is that girls, and boys from homes with educated parents, are given more encouragement by their teachers. The point here is that “(girls and boys) from homes with educated parents” is not the same as “girls (and boys from homes with educated parents).”

2. When the associative law holds, our work is simplified. No matter how many objects are to be combined, once the left-to-right order of the objects has been set, no matter how they are paired off and operated on, the result will be the same. Therefore, it is not necessary to use parentheses to give the operation an unambiguous meaning, and that makes for easier reading.

distributive law, in mathematics, the law relating the operations of multiplication and addition, stated symbolically, $a(b + c) = ab + ac$; that is, the monomial factor a is distributed, or separately applied, to each term of the binomial factor $b + c$, resulting in the product $ab + ac$. From this law it is easy to show that the result of first adding several numbers and then multiplying the sum by some number is the same as first multiplying each separately by the number and then adding the products.

In abstract algebra and formal logic, the distributive property of binary operations generalizes the distributive law from elementary algebra. In propositional logic, distribution refers to two valid rules of replacement. The rules allow one to reformulate conjunctions and disjunctions within logical proofs.

For example, in arithmetic:
$2 \cdot (1 + 3) = (2 \cdot 1) + (2 \cdot 3)$, but $2 / (1 + 3) \neq (2 / 1) + (2 / 3)$.

In the left-hand side of the first equation, the 2 multiplies the sum of 1 and 3; on the right-hand side, it multiplies the 1 and the 3 individually, with the products added afterwards. Because these give the same final answer (8), we say that multiplication by 2 distributes over addition of 1 and 3. Since we could have put any real numbers in place of 2, 1, and 3 above, and still have obtained a true equation, we say that multiplication of real numbers distributes over addition of real numbers.

Commutative Law of Multiplication

Commutative Law of Multiplication is a fancy way of saying when you multiply two numbers, it doesn't matter which number you put down first and which number you put down second.

$$a * b = b * a$$

This basic law of arithmetic is taught in the second grade in elementary school. Yet it is very useful when you evaluate the relative merits between Traditional 401K, ROTH IRA, and the new Roth 401k.

Blogger Trent writes the popular blog The Simple Dollar, which is one of the most successful personal finance blogs. Unfortunately Trent made the mistake of not recognizing the Commutative Law of Multiplication. In his post The New Roth 401(k) Versus The Traditional 401(k): Which Is The Better Route? he said Roth 401k is better even if the tax rate in the future is lower than the tax rate at present. His reasoning was

"Basically, by paying $2,800 a year now in extra taxes, Joe saves himself $14,000 a year in retirement."

Wrong. It matters not how much tax you pay at different times. What matters is how much money you have left after all the taxes are paid. Sadly when more than one commenters pointed out the problem with Trent's math, he still insisted that his math was correct.

In case someone out there is still confused, here's how the math works. Let $t0$ be the marginal tax rate now, and $t1$ be the marginal tax rate at retirement time. Suppose through successful INVESTING, you are able to grow each dollar to $\$n$ when you are ready to retire. For each dollar you invest in a Traditional 401K, you will have $\$n$ before tax, and $n * (1 – t1)$ after tax. In a ROTH IRA or Roth 401k, for each dollar before tax, you pay tax first and have $(1 – t0)$ dollars left after tax. Growing the money to the same degree, you will have $(1 – t0) * n$ when you are ready to retire. If the tax rate now ($t0$) is the same as the tax rate at retirement time ($t1$), we have

$$n * (1 – t1) = (1 – t0) * n$$

There, is the Commutative Law of Multiplication.

If the tax rate at retirement time is lower, t1 < t0, Traditional 401k will be better than Roth 401k because the value on the left hand side is larger than the value on the right hand side. The opposite is true if the tax rate at present is lower, t0 < t1.

Of course nobody knows what the future tax rates will be or whether they will be higher or lower than today’s. In choosing between a Traditional 401k and a Roth 401k, you just have to take a guess or do a little of both. For me, my money is on the Traditional 401k. I think the Roth 401k is a device for the current government to maximize its current revenue at the cost of robbing revenues from the future government. When the future government needs money, it will find ways to raise revenue including taxing on Roth withdrawals either directly or indirectly. The laws on Roth IRA and Roth 401k only say withdrawals from them today are not taxed. They don’t say withdrawals won’t ever be taxed. Tax laws can be changed by the legislature in the future.